

Multistability and stochastic phenomena in a randomly forced thermochemical system

I Bashkirtseva¹, A N Pisarchik² and L Ryashko¹

¹ Ural Federal University, Lenina, 51, Ekaterinburg, 620000, Russia

² Center for Biomedical Technology, Technical University of Madrid, 28223 Pozuelo de Alarcon, Madrid, Spain

E-mail: lev.ryashko@urfu.ru

Abstract. A model of the thermochemical reactor with well mixing is considered. An impact of random noise on the attractors of this model is studied in zones of saddle-node bifurcations with transitions from mono- to bimodal dynamics. Phenomena of the stochastic excitability and noise-induced transitions are demonstrated. Critical values of the noise intensities corresponding to the onset of generation of the large-amplitude stochastic oscillations are estimated.

1. Introduction

It is well known that in nonlinear systems even a weak noise can generate the wide variety of stochastic phenomena [1-4]. Such effects as the stochastic resonance, noise-induced chaos, stochastic bifurcations are observed in various nonlinear models in different domains of science [5-9]. In particular, the mathematical models of chemical kinetics [10, 11] demonstrate the complex dynamic regimes extremely sensitive to noise. A nonlinear model of thermochemical reactor [12-14] is one of such examples with excitable stochastic dynamics.

In the present paper, we study this model in zones of saddle-node bifurcations where the transitions from mono- to bimodal behavior are observed. An impact of random noise on the attractors of this model is investigated. We focus on two variants of the saddle-node bifurcation when the additional attractor (equilibrium or cycle) appears.

In Section 2, the dynamic regimes of the deterministic model of the thermochemical reactor with well mixing are shortly discussed.

In Section 3, an influence of random noise on attractors of this model is studied both in the mono- and bistable zones. In zones of stable equilibria near the saddle-node bifurcation, a phenomenon of the stochastic excitability is shown. We demonstrate how increasing noise transforms the system to the regime of large-amplitude stochastic oscillations. The noise-induced transitions between coexisting attractors (equilibrium–equilibrium and equilibrium–cycle) are discussed.

2. Deterministic thermochemical model

Consider a deterministic system

$$\begin{aligned} \dot{x} &= \sqrt{y} \left(-x \exp \left(-\frac{\delta}{y} \right) + p(1-x) \right) \\ \dot{y} &= \frac{2}{3} q \sqrt{y} \left(x \exp \left(-\frac{\delta}{y} \right) + r(1-y) \right), \end{aligned} \quad (1)$$



where x and y are the dimensionless concentration and temperature, respectively; parameters p , q , r and δ are positive. The system (1) models the dynamic processes in the thermochemical reactor with well mixing [12].

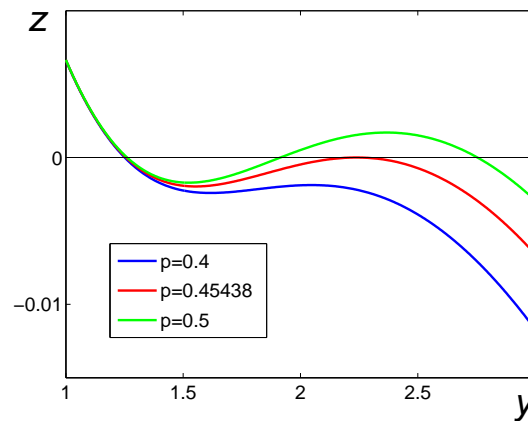


Figure 1. Plots of the function $z(y)$.

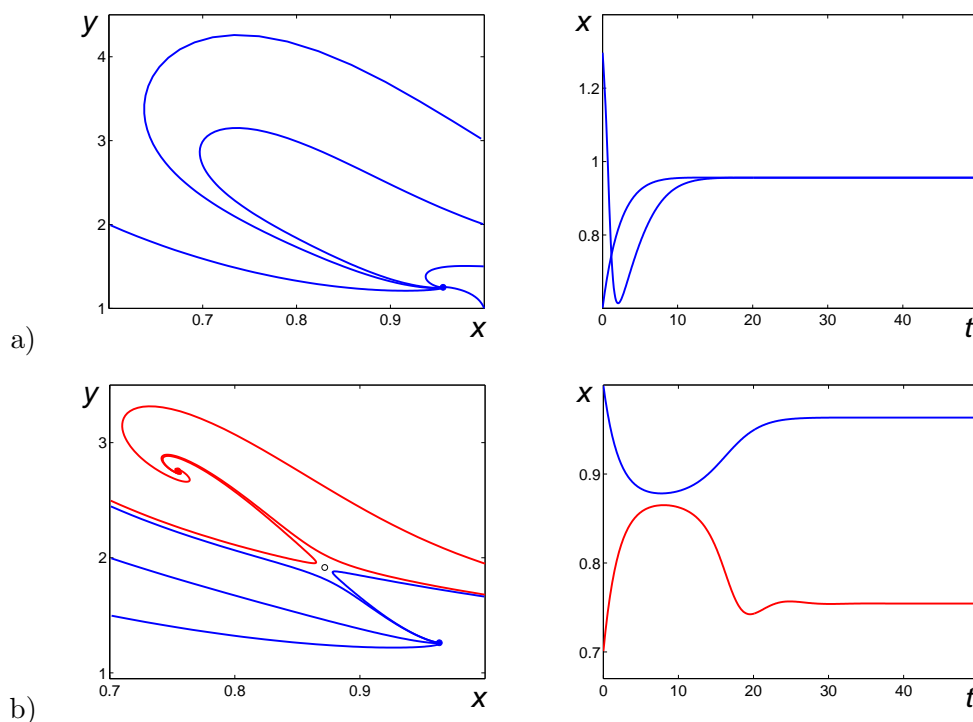


Figure 2. Phase trajectories and time series of deterministic system (1) with $\delta = 5$, $r = 0.07$, $q = 40$ and a) $p = 0.4$, b) $p = 0.5$.

In the present paper, we study a behavior of this system in zones of the transition from mono- to bistability. Here, we consider a case when an appearance of the additional attractor

occurs as a result of the saddle-node bifurcation. In system (1), two variants of the saddle-node bifurcation can be observed. One of them (variant A) is connected with the birth of the pair of the stable and unstable equilibria. The second one (variant B) is a result of the appearance of stable and unstable cycles.

At first, consider the variant A of the saddle-node bifurcation for the fixed $\delta = 5$, $r = 0.07$, $q = 40$ and varying p .

The coordinates of equilibria of model (1) satisfy the following system

$$z(y) = 0, \quad x = \frac{p}{p + \exp\left(-\frac{\delta}{y}\right)},$$

where

$$z(y) = \frac{p}{p \exp\left(\frac{\delta}{y}\right) + 1} + r(1 - y).$$

In Fig. 1, plots of the function $z(y)$ are shown for three values of parameter p . As can be seen, system (1) has a single equilibrium for $p = 0.4$, and has three equilibria for $p = 0.5$. The value $p_* = 0.45438$ where the plot of $z(y)$ is tangent to the y -axis corresponds to the saddle-node bifurcation. In Fig. 2, the phase portraits and time series of system (1) for $p = 0.4$ and $p = 0.5$ are presented.

The system (1) with $p = 0.4$ is monostable: all trajectories tend to the single stable equilibrium $\bar{x} = 0.9561$, $\bar{y} = 1.2508$ (see Fig. 2a).

For $p = 0.5$, system (1) is bistable with two stable equilibria $M_1(0.7544, 2.7544)$ and $M_2(0.9634, 1.2612)$, separated by the saddle equilibrium $M_3(0.8719, 1.9153)$ (see Fig. 2b). The basins of attraction of equilibria M_1 and M_2 are separated by the stable manifold of the saddle M_3 .

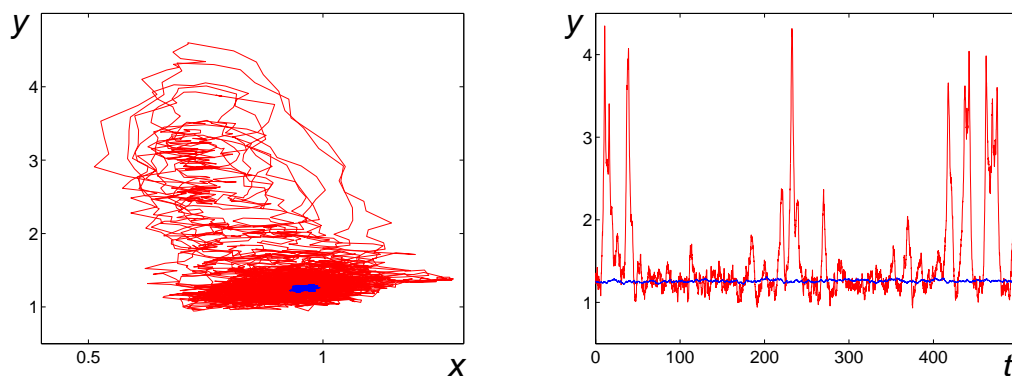


Figure 3. Phase trajectories and time series of stochastic system (2) with $\delta = 5$, $r = 0.07$, $q = 40$, $p = 0.4543$ for $\varepsilon = 0.01$ (blue) and $\varepsilon = 0.1$ (red).

3. Stochastic model

To study an impact of random disturbances on the dynamics of the thermochemical system, we will use the following stochastic model:

$$\begin{aligned} \dot{x} &= \sqrt{y} \left(-x \exp\left(-\frac{\delta}{y}\right) + p(1 - x) \right) + \varepsilon_1 \xi_1(t), \\ \dot{y} &= \frac{2}{3} q \sqrt{y} \left(x \exp\left(-\frac{\delta}{y}\right) + r(1 - y) \right) + \varepsilon_2 \xi_2(t). \end{aligned} \quad (2)$$

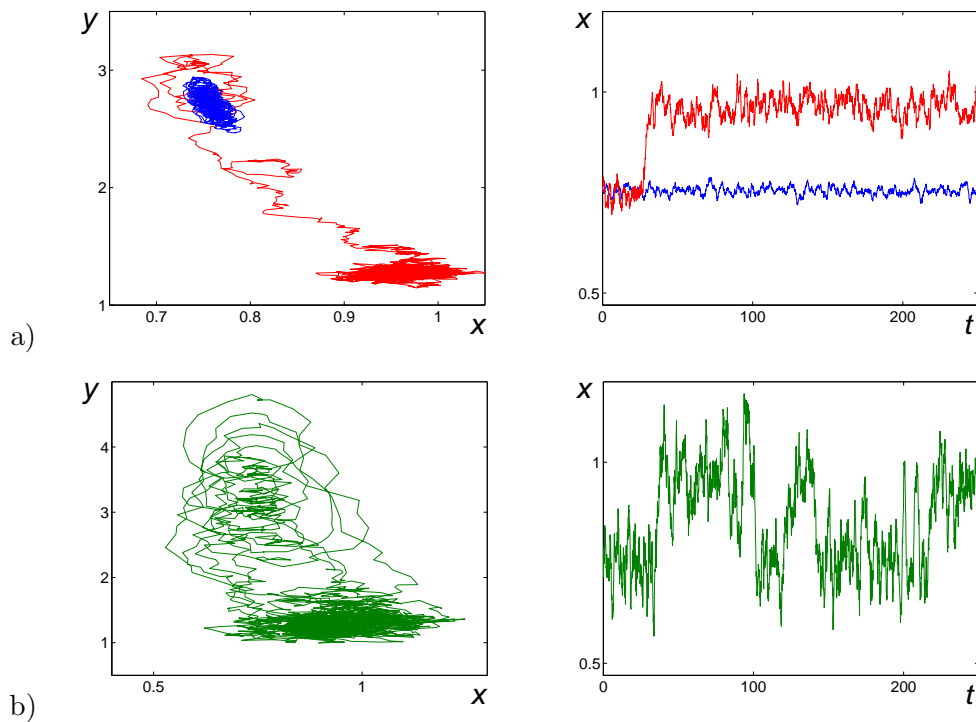


Figure 4. Phase trajectories and time series of the stochastic system (2) with $\delta = 5$, $r = 0.07$, $q = 40$, $p = 0.5$ for a) $\varepsilon = 0.01$ (blue), $\varepsilon = 0.03$ (red); b) $\varepsilon = 0.08$ (green).

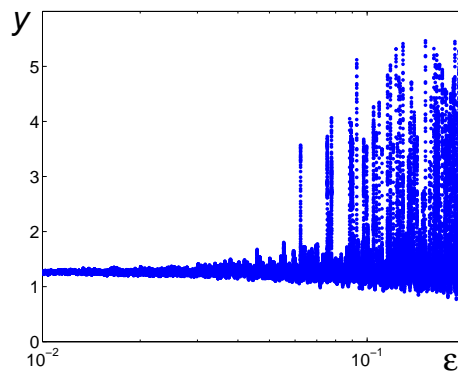


Figure 5. Random states of the stochastic system (2) $\delta = 5$, $r = 0.07$, $q = 40$, $p = 0.5$.

Here, $\xi_{1,2}(t)$ are the standard Gaussian uncorrelated processes, and $\varepsilon_{1,2}$ are the noise intensities. In what follows, we put $\varepsilon_1 = \varepsilon_2 = \varepsilon$.

3.1. Stochastic excitability

Consider an influence of noise on the monostable system with $p = 0.4543$ that is close to the saddle-node bifurcation point p_* . In Fig. 3, for two values of the noise intensity, the random trajectories of system (2) starting from the stable equilibrium are shown.

For a weak noise ($\varepsilon = 0.01$), the random trajectories are localized in a small vicinity of the

equilibrium, and system (2) exhibits the small-amplitude stochastic oscillations (SASO). When the noise intensity increases and exceeds some threshold value, along with SASO, the large-amplitude stochastic oscillations (LASO) appear (see Fig. 3 for $\varepsilon = 0.1$). So, an alternation of SASO and LASO specifies a stochastic excitability of the stochastic system (2) close to p_* .

3.2. Bistability and noise-induced transitions

Consider now an impact of noise on the system in the bimodal zone where two stable equilibria M_1 and M_2 coexist.

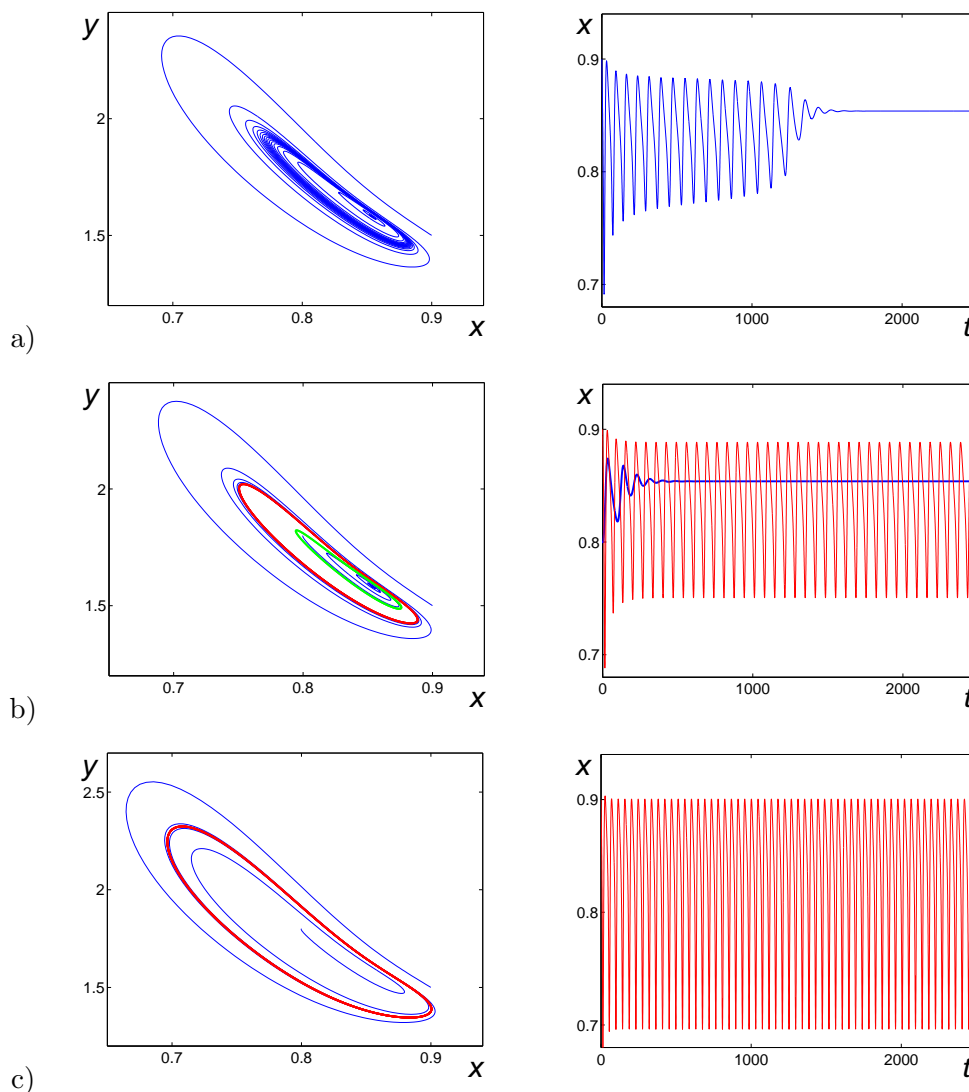


Figure 6. Phase trajectories and time series of the deterministic system (2) with $\delta = 5$, $r = 0.0622$, $p = 0.2475$ for a) $q = 39.6$, b) $q = 40$, c) $q = 43.5$.

In Fig. 4, for three values of the noise intensity, the random trajectories of system (2) starting from the stable equilibrium M_1 are shown.

For a weak noise ($\varepsilon = 0.01$), the random trajectories are still reside in a small vicinity of the equilibrium M_1 . When the noise intensity increases, a transition from the basin of attraction of

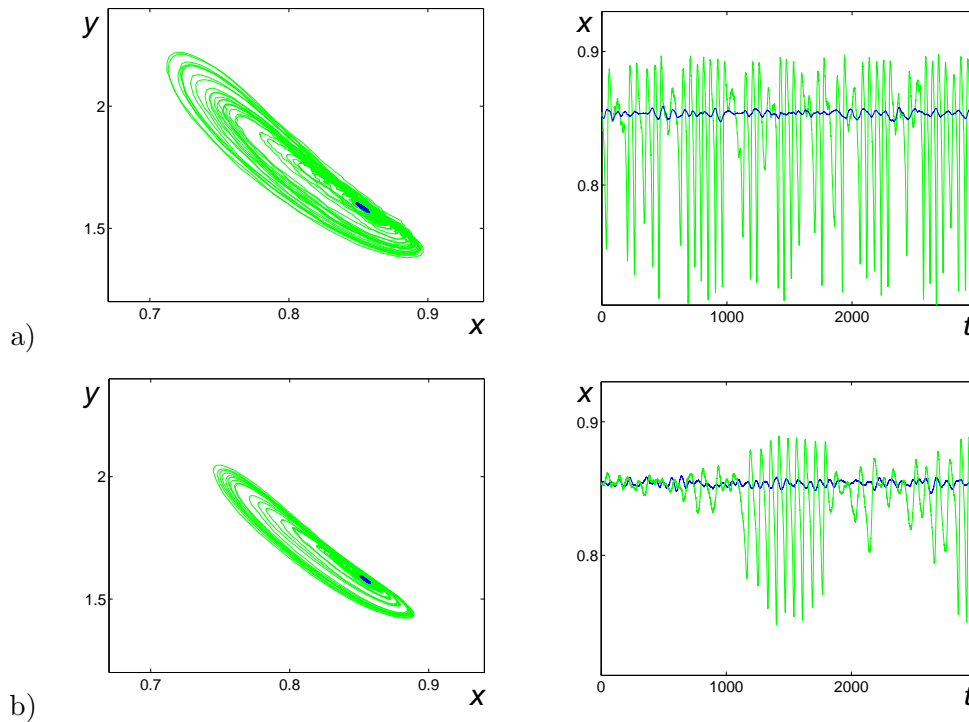


Figure 7. Phase trajectories and time series of the stochastic system (2) with $\delta = 5$, $r = 0.0622$, $p = 0.2475$ for a) $q = 39.6$ and $\varepsilon = 0.0001$ (blue), $\varepsilon = 0.001$ (green); b) $q = 40$ and $\varepsilon = 0.0001$ (blue), $\varepsilon = 0.0003$ (green).

M_1 to M_2 occurs (see Fig. 4a for $\varepsilon = 0.03$). So, the noise induces a transition of the considered thermochemical system to another alternative noisy equilibrium regime.

A further increase of noise generates the repetitive transitions between basins of attraction of M_1 and M_2 . As a result, the large-amplitude oscillations are observed (see Fig. 4b for $\varepsilon = 0.08$). Some details of such changes in stochastic dynamics are well seen in Fig. 5 where the random states of the stochastic system (2) versus noise intensity ε are plotted. Here, the critical values of noise intensities corresponding to the onset of generation of the large-amplitude stochastic oscillations can be estimated.

3.3. Stochastic dynamics near the bifurcation of the cycle birth

Consider now the variant B of saddle-node bifurcation with the appearance of stable and unstable cycles.

Following [14], we put $\delta = 5$, $r = 0.0622$, $p = 0.2475$ and consider the dynamics depending on parameter q .

In Fig. 6, for three values of parameter q , the phase portraits and time series of deterministic system are shown. For $q = 39.6$, the system is monostable with a stable focus as a single attractor, and all trajectories tend to it (see Fig. 6a). For $q = 40$, the system is bistable and demonstrates a coexistence of the stable equilibrium and stable cycle. Their basins of attraction are separated by the unstable cycle (see Fig. 6b). For $q = 43.5$, the system is again monostable with a stable cycle as a single attractor (see Fig. 6c).

It is interesting that near the saddle-node bifurcation, in the zone of monostability where a stable equilibrium is a single attractor, system (2) is excitable as well. Indeed, for a small noise,

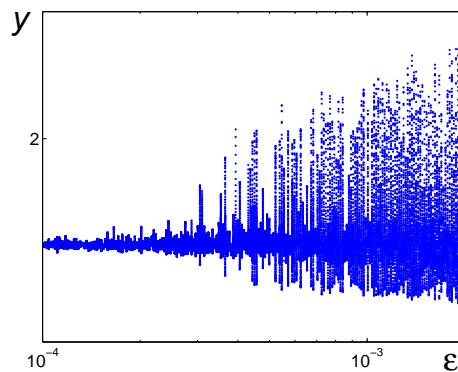


Figure 8. Random states of the stochastic system (2) with $\delta = 5$, $r = 0.0622$, $p = 0.2475$, $q = 39.6$.

system (2) exhibits SASO with random trajectories located near the stable equilibrium (see Fig. 7a) for $\varepsilon = 0.0001$). With increasing noise, an alternation of SASO and LASO is observed (see Fig. 7a) for $\varepsilon = 0.001$).

Such alternation in the zone of bistability (see Fig. 7b) is also observed.

Some details of such noise-induced transitions to LASO are shown in Fig. 8 where the random states of stochastic system (2) versus noise intensity ε are plotted for $q = 39.6$.

As a conclusion, we have to note that the nonlinear model under consideration demonstrates a wide variety of noise-induced effects which must be taken into consideration in the engineering problems.

4. References

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Acknowledgments

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